Two Theorems Dealing With Bounds on the Magnitudes of the Distances Between the n-Dimensional Cashwell-Everett Means

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Continuing to develop the theory of weighted n-dimensional means, this article presents two theorems that follow Theorem 1 (Glaser, 2002) and, hence, are labeled Theorems 2 and 3. Theorem 2 gives a bound on the distance between two one dimensional weighted means (Cashwell-Everett, 1969). Theorem 3 generalizes this bound for the distance between two n-dimensional means.

We recall Theorem 1 (Glaser, 2002) which gave bounds on the defining functions \( A_k(x_k), B_k(x_k), C_k(x) \) and \( D_k(x_k) \).

**Theorem 1.** Under the same hypotheses as in the n-dimensional deviation theorem (Glaser, 2002), the coordinates \( b_k \) and \( c_k \) of the means generated by \( b_k(x_k) \) and \( c_k(x_k) \) respectively satisfy the following inequalities

\[
0 < -C_k(b_k) < -D_k(b_k) < \left[ \frac{1}{g_k(x_k)} - \frac{1}{g_k(x_{k0})} \right] \min\{-B_k(x_k), B_k(x_{k0})\}
\]

\[
0 < B_k(c_k) < A_k(c_k) < \left[ \frac{1}{g_k(x_k)} - \frac{1}{g_k(x_{k0})} \right] \min\{-C_k(x_k), C_k(x_{k0})\}
\]

**Theorem 2. Part I.** The mean \( b_k \) of \( x_k(\vec{v}_m) \) relative to \( \mathcal{W}_k(\vec{b}_k, \vec{v}_m) \) satisfies the inequality

\[
0 < c_k - b_k < \left[ \frac{1}{g_k(x_k)} - \frac{1}{g_k(x_{k0})} \right] \min\{-B_k(x_k), B(x_{k0})\} \int_{\mathcal{W}_k(\vec{b}_k, \vec{v}_m)} d\mathcal{W}_m
\]

where \( \vec{b}_k \in (b_k, c_k) \).

**Part II.** The mean \( c_k \) of \( x_k(\vec{v}_m) \) relative to \( g_k(\vec{b}_k, \vec{v}_m) \) satisfies the inequality

\[
0 < c_k - b_k < \left[ \frac{1}{g_k(x_k)} - \frac{1}{g_k(x_{k0})} \right] \min\{-C_k(x_k), C(x_{k0})\} \int_{\mathcal{W}_k(\vec{b}_k, \vec{v}_m)} d\mathcal{W}_m
\]

where \( \vec{b}_k \in (b_k, c_k) \), and \( \mathcal{W}_m = \left\{ (v_i, \ldots, v_m) | v_j^l < v_j < v_j^l, v_j^l \in J \right\} \).

**Proof:**

**Part I:** Since \( C_k(x_k) \) is continuous on the closed interval \([b_k, c_k]\) and differentiable on the open interval \((b_k, c_k)\), the mean value theorem implies there exists \( \vec{b}_k \in (b_k, c_k) \) such that

\[
C_k(c_k) - C_k(b_k) = C_k'(\vec{b}_k) \bullet (c_k - b_k)
\]

but \( C_k(c_k) = 0 \) hence, \( -C_k' = C_k'(\vec{b}_k) \bullet (c_k - b_k)\).

Part I of theorem 1 now implies that
\[ C_k (\bar{g}_k) \left( \bar{b}_k - b_k \right) < \left[ g_k (x_{k \rho}) - g_k (x_{k \omega}) \right] \min \left\{ -B_k (x_{k \rho}), B_k (x_{k \omega}) \right\} \]  \hspace{1cm} (3)

On the other hand, we also have \[ C_k (\bar{g}_k) = \int_{\mathbb{R}^n} g_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) dV_m \] and since \( b_k < x_k \) and \( g_k \) is monotone increasing, we find \( g_k (b_k) \int_{\mathbb{R}^n} W_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) dV_m < C_k (\bar{g}_k) \).  \hspace{1cm} (4)

Inequalities (3) and (4) now imply:

\[ \left[ g_k (b_k) \int_{\mathbb{R}^n} W_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) dV_m \right] (c_k - b_k) < \left[ g_k (x_{k \rho}) - g_k (x_{k \omega}) \right] \min \left\{ -B_k (x_{k \rho}), B_k (x_{k \omega}) \right\} \]

which in turn yields inequality (1).

**Part II.** Again by the mean value theorem there exists \( \bar{g}_k (x_{k \rho}, \bar{v}_m) \) such that \( B_k (c_k) = B_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) \) since \( B_k (b_k) = 0 \).

Part II of Theorem 1 can now be written as

\[ B_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) \bullet (c_k - b_k) < \left[ \frac{1}{g_k (x_{k \rho})} - \frac{1}{g_k (x_{k \omega})} \right] \min \left\{ -C_k (x_{k \rho}), C_k (x_{k \omega}) \right\} \]  \hspace{1cm} (5)

On the other hand also \( B_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) = \int_{\mathbb{R}^n} W_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) dV_m \) and, therefore, (5-23) becomes

\[ \left[ \int_{\mathbb{R}^n} W_k (\bar{g}_k (x_{k \rho}, \bar{v}_m)) dV_m \right] (c_k - b_k) < \left[ \frac{1}{g_k (x_{k \rho})} - \frac{1}{g_k (x_{k \omega})} \right] \min \left\{ -C_k (x_{k \rho}), C_k (x_{k \omega}) \right\} \]

which then yields inequality (2).

QED

We next use these results to obtain also two bounds for the distance between the \( n \)-dimensional means \((b_1, \ldots, b_n)\) and \((c_1, \ldots, c_n)\) defined by the functions.

\[ B(x_1, \ldots, x_n) = \prod_{k=1}^{n} \int_{\mathbb{R}^n} W_k (\bar{g}_k, \bar{v}_m) d\xi_k dV_m \] and

\[ C(x_1, \ldots, x_n) = \prod_{k=1}^{n} \int_{\mathbb{R}^n} W_k (\bar{g}_k, \bar{v}_m) d\xi_k dV_m \] respectively.

**Theorem 3.** The distance in Euclidean \( n \)-space between the means \((b_1, \ldots, b_n)\) and \((c_1, \ldots, c_n)\) satisfies the inequalities

\[ \sqrt{n} \left( \sum_{k=1}^{n} (c_k - b_k) \right)^2 < \sqrt{n} \left( \frac{\max_{1 \leq k < n} g_k (x_{k \rho}) - \min_{1 \leq k \leq n} g_k (x_{k \omega})}{\max_{1 \leq k < n} \left\{ -C_k (x_{k \rho}), C_k (x_{k \omega}) \right\}} \right) \min_{1 \leq k \leq n} g_k (b_k) \bullet \inf_{\bar{g}_k} W_k (\bar{g}_k, \bar{v}_m) dV_m \]  \hspace{1cm} (6)

and
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\[
\sum_{k=1}^{n} (c_k - b_k)^2 < \sqrt{n} \left[ \min_{1 \leq k < n} \frac{1}{g_k(x_{kl}) - g_k(x_{kl})} \right] \left( \max_{1 \leq k < n} \frac{1}{g_k(x_{kl}) - g_k(x_{kl})} \right) \max_{1 \leq k < n} \left[ \min_{1 \leq k < n} \left[ -c_k(x_{kl}), c_k(x_{kl}) \right] \right] \min_{1 \leq k < n} \left[ \inf_{1 \leq k < n} \left[ -c_k(x_{kl}), c_k(x_{kl}) \right] \right] \nu \Omega \nu
\]

(7)

where \( R_{m,k} = [x_{kl}, x_{kl}] \times \Omega \).

We can calculate the volume of \( \nu \Omega \nu \) as follows:

\[
\nu \Omega \nu = \int_{\nu} d\nu = \int_{\nu} h(\nu) \prod_{k=1}^{n} \nu_k(\nu_k) d\nu
\]

\[
= \frac{\partial}{\partial \nu} \prod_{k=1}^{n} \nu_k(\nu_k) \prod_{k=1}^{n} \nu_k(\nu_k) d\nu
\]

Proof:

Part I: Clearly

\[
[g_k(x_{kl}) - g_k(x_{kl})] \min_{1 \leq k < n} \left[ -B_k(x_{kl}), B_k(x_{kl}) \right] < \max_{1 \leq k < n} \left[ g_k(x_{kl}) - g_k(x_{kl}) \right] \max_{1 \leq k < n} \left[ \min_{1 \leq k < n} \left[ -B_k(x_{kl}), B_k(x_{kl}) \right] \right]
\]

and also

\[
g_k(b_k) \int_{\nu} \nu_k(\nu_k) d\nu > \min_{1 \leq k < n} \left[ g_k(b_k) \right] \inf_{1 \leq k < n} [\nu_k(\nu_k)] \int_{\nu} d\nu
\]

Theorem 2 can now be applied to obtain

\[
\sum_{k=1}^{n} (c_k - b_k)^2 < \left[ \sum_{k=1}^{n} \left[ g_k(x_{kl}) - g_k(x_{kl}) \right] \min_{1 \leq k < n} \left[ -B_k(x_{kl}), B_k(x_{kl}) \right] \right] ^{1/2}
\]

and using (8) and (9), we find

\[
\sum_{k=1}^{n} (c_k - b_k)^2 < \left[ \max_{1 \leq k < n} \left[ g_k(x_{kl}) - g_k(x_{kl}) \right] \max_{1 \leq k < n} \left[ \min_{1 \leq k < n} \left[ -B_k(x_{kl}), B_k(x_{kl}) \right] \right] \right] ^{1/2}
\]

from which (6) follows.

Part II. The proof of inequality (7) is similar and uses the inequality:

\[
- \frac{1}{g_k(x_{kl})} - \frac{1}{g_k(x_{kl})} \min_{1 \leq k < n} \left[ -c_k(x_{kl}), c_k(x_{kl}) \right] < \frac{1}{g_k(x_{kl})} - \frac{1}{g_k(x_{kl})} \max_{1 \leq k < n} \left[ \min_{1 \leq k < n} \left[ -c_k(x_{kl}), c_k(x_{kl}) \right] \right]
\]
and also the fact that $\int_{x_{k2}}^{x_{k1}} \nu^2 \left( \frac{e^{x_{k2}} e^{x_{k1}}}{\nu^2_m} \right) d\nu_m > \min_{k} \left[ \inf_{k} \left( \frac{x_{k2}}{x_{k1}} \nu^2_m \right) \right] 2 \nu_{1} \nu_{2}$. This proves Theorem 3.

An example illustrating Theorem 3: Let $\xi_{k}(\nu_{1}) = \nu_{1}^2$ and $\eta_{k}(\nu_{1}) = \nu_{1}$ where $\nu_{1} \in \mathbb{R} \setminus [0,1]$ then $\inf_{k} (\nu_{1}) = 0$ and $\sup_{k} (\nu_{1}) = 1$. Let $x_{k1}(\nu_{2}) = x_{k}(\nu_{1}, \nu_{2}) = \nu_{2}$ for $k = 1,2$ be two non-constant functions defined on $x_{k1} = [x_{k1}, x_{k2}] \times [0,1]$ and $x_{k2} = 1$. Clearly $x_{k1} = 0$ and $x_{k2} = 1$ for $k = 1,2$.

Define the two positive continuous weight functions $\nu_{k}(x_{k}, \nu_{1}, \nu_{2}) = 1 + \nu_{1} + \nu_{2}$ on the Cartesian product $x_{k1} \times x_{k2} \times [0,1]$. The two functions have been chosen equal in this example in order to obtain symmetry. Let $g_{k}(x_{k}) = x_{k}(k = 1,2)$ be defined on $[x_{k1}, x_{k2}] = [0,1]$. These two functions are continuous, strictly increasing and positive almost everywhere on $[x_{k1}, x_{k2}] = [0,1]$. Define the four functions:

\[
B_{k}(x_{k}) = \int_{x_{k1}}^{x_{k2}} \int_{x_{k1}}^{x_{k2}} (1 + \nu_{1} + \nu_{2}) dx_{k1} dx_{k2} d\nu \nu_{1} \\
C_{k}(x_{k}) = \int_{x_{k1}}^{x_{k2}} \int_{x_{k1}}^{x_{k2}} \nu_{1} dx_{k1} dx_{k2} d\nu \nu_{1}
\]

where $k = 1,2$. Integrating, we find $B_{k}(x_{k}) = \frac{1}{60} (19x_{k} - 21) \cdot C_{k}(x_{k}) = \frac{1}{120} (19x_{k}^2 - \frac{47}{7})$. Solving $B_{k}(b_{k}) = 0$, we find $b_{k} = \frac{21}{38} \approx 0.55$. Solving $C_{k}(c_{k}) = 0$, we find $c_{k} = \frac{47}{133} \approx 0.59$. The left hand side of inequality (6) becomes $\frac{\sum_{k=1}^{2} (c_{k} - b_{k})^2}{\sum_{k=1}^{2}} = \sqrt{2}(0.59 - 0.55) \approx 0.0566$ since $n = 2$.

To compute the right hand side of inequality (6), we find the following:

$$\max_{k=1,2} g_{k}(x_{k}) = 1, \quad \min_{k=1,2} g_{k}(x_{k}) = 0$$

$$\min_{k=1,2} \left\{ -C_{k}(x_{k}), C_{k}(x_{k}) \right\} = \min_{k=1,2} \left\{ -C_{k}(x_{k}), C_{k}(x_{k}) \right\} = \frac{47}{840}$$

Hence, $\max_{k=1,2} \left\{ \min_{k=1,2} \left\{ -C_{k}(x_{k}), C_{k}(x_{k}) \right\} \right\} = \frac{47}{840}$

Also $g_{k}(b_{k}) = b_{k} = 0.5$ \quad $C_{k} = 1,2$

Hence, $\min_{k=1,2} g_{k}(b_{k}) = 0.55$

$$\inf_{k=1,2} \nu_k((\xi_{k1}, \xi_{k2})) = \inf_{\nu_1, \nu_2 \in [1]} (1 + \nu_1 + \nu_2) = 1$$

Finally, $\nu_{1} \nu_{2} = \int_{0}^{1} (\nu_{1} \nu_{2}) d\nu_{1} \nu_{2} = \frac{1}{6}$; therefore, the right hand side of inequality (6) becomes:
\[
\frac{\sqrt{3}(1 - 0)}{\frac{47}{840}} \approx 0.1439.
\]

In numbers, inequality (6) becomes 0.05656 < 0.1439. In this example, \( \min_{k=1,2} \varrho_k(x_{k_i}) = 0 \) therefore, inequality (7) is the trivial statement 0.05656 < \( \infty \).

**CONCLUSION**

In this article, we have established that the bounds on the distance between means of the defining functions \( B_k(x_k) \) and \( C_k(x_k) \) and therefore also of the means of the defining functions

\[
B(\bar{x}_n) = \prod_{k=1}^{n} B_k(x_k)
\]

and

\[
C(\bar{x}_n) = \prod_{k=1}^{n} C_k(x_k)
\]

in \( n \)-dimensional space. A simple example that was designed to avoid huge computations has also been included. The next topic in the theory of generalized means deals with separable weight functions which leads to many interesting results.

**REFERENCES:**

Cashwell, E. C. and Everett, J. C., (1968). The mean of a function \( x(v) \) relative to a weight function \( w(\xi, v) \).
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