THE REPEATED SUMS OF INTEGERS

A Thesis
Presented to the
Faculty of
California State Polytechnic University, Pomona

In Partial Fulfillment
Of the Requirements for the Degree
Master of Science
In
Mathematics

By
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2015
SIGNATURE PAGE

THESIS: THE REPEATED SUMS OF INTEGERS

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DATE SUBMITTED: Spring 2015

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ACKNOWLEDGMENTS

I would like to say thanks to my advisor, Dr. Mihaila. She has been a great help for me not only for the course of my thesis research, but also for my whole journey in Mathematics. Dr. Mihaila was the one to show me that I have the potential to pursue a career in Math. I would also like to thank all of my Math professors. Each and everyone of you have shown me a different side of Mathematics, and for the most part I have enjoyed them all.

Lastly, I would like to say thank to my girlfriend, Trang. She has always been there for me, through ups and downs. I would not be here today without her.
ABSTRACT

It is well-known that the sum of integers from 1 to $n$ is $\frac{n(n + 1)}{2}$. But what happens when we add these sums together? Do we have a closed form formula for $\sum_{i=1}^{n} \sum_{j=1}^{i} j$? What happens if we keep repeating the process? In general, do we have a closed form formula for $\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} ... \sum_{a_n=1}^{a_{n-1}} a_n$? We will then look at the sum of squares, cubes,... of consecutive integers and repeat the process. Would we always get a nice closed form formula for the finite multiple sum of powers of integers? Are there other sequences that we can compute the partial multiple sum? In this thesis we will discover it all.
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Chapter 1

Introduction

It is commonly known that the sum $1 + 2 + 3 + ... + n$ is equal to $\frac{n(n+1)}{2}$. This sum is often seen in the first course of logic and/or discrete mathematics and should be familiar to every mathematician. But what would happen if we take the double sum of consecutive integers? In other words, do we have a formula for $\sum_{i=1}^{n} \sum_{j=1}^{i} j$? Furthermore, can we put finitely many more sum notations in front of it and expect to get a nice formula? In this first chapter, we will try to find a closed form formula of the double sums, the triple sums, and in general, the finitely many sums of consecutive integers from 1, $\sum_{a_{1}=1}^{n} \sum_{a_{2}=1}^{a_{1}} ... \sum_{a_{j}=1}^{a_{j-1}} a_{j}$.

**Theorem 1.0.1.** For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$.

**Proof.** Let $S = 1 + 2 + 3 + ... + n$, then we can write:

$$2S = 1 + 2 + 3 + ... + n$$

$$+ n + (n-1) + (n-2) + ... + 1$$

$$= (n+1) + (n+1) + (n+1) + ... + (n+1)$$

Or $2S = n(n + 1)$, and therefore $S = \frac{n(n + 1)}{2}$.
Theorem 1.0.2. For all $n \in \mathbb{N}$, $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$.

Proof. We will prove by induction. Let $P(n)$ be the statement that for $n \geq 1$, $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n + 1)(2n + 1)$. First we check $P(1): \sum_{k=1}^{n} k^2 = 1^2 = 1$ and $\frac{1}{6}1(1 + 1)(2 + 1) = 1$. So $P(1)$ is true. Suppose $P(k)$ is true for some $k \geq 1$, that is, $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k + 1)(2k + 1)$. We will show $P(k + 1)$ is also true. Now $\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k + 1)^2$. Applying the induction hypothesis, we can write:

$$\sum_{i=1}^{k+1} i^2 = \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$$

$$= \frac{(k(k + 1)(2k + 1) + 6(k + 1)^2}{6}$$

$$= \frac{(k + 1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k + 1)(k + 2)(2(k + 1) + 1)}{6}$$

Therefore $P(k + 1)$ is also true and we get the desired result. \qed

Theorem 1.0.3. For all $n \in \mathbb{N}, n \geq 1$, $\sum_{i=1}^{n} \sum_{k=1}^{i} k = \frac{1}{6}n(n + 1)(n + 2)$.

Proof. By Theorem 1.0.1 $\sum_{k=1}^{n} k = \frac{1}{2}n(n + 1)$, and by Theorem 1.0.2 we can write

$$\sum_{i=1}^{n} \sum_{k=1}^{i} k = \sum_{i=1}^{n} \frac{1}{2}i(i + 1)$$

$$= \frac{1}{2} \sum_{i=1}^{n} (i^2 + i)$$

$$= \frac{1}{2} \left[ \left( \frac{1}{6}n(n + 1)(2n + 1) \right) + \frac{1}{2}n(n + 1) \right]$$

$$= \frac{1}{6}n(n + 1)(n + 2).$$

\qed
We can continue to use basic algebra to find a formula for \( \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} k \), or in general, \( \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j \). But the more summations we add, the messier it will get. Instead, we will need a stronger tool to come up with a generalization. The tool we will need is combinatorics.

**Definition 1.0.1.** For natural numbers \( n \) and \( k \) such that \( n \geq k \), the binomial coefficient \( \binom{n}{k} \) is the number of ways of picking \( k \) unordered outcomes from \( n \) possibilities. The binomial coefficient is given by the formula \( \binom{n}{k} = \frac{n!}{k!(n-k)!} \).

**Theorem 1.0.4.** For all \( n \in \mathbb{N} \), \( \sum_{k=1}^{n} k = \frac{n+1}{2} \) and \( \sum_{i=1}^{n} \sum_{k=1}^{i} k = \frac{n+2}{3} \).

**Theorem 1.0.5.** For all \( n, k \in \mathbb{N} \) such that \( n \geq k \), \( \binom{n}{k} = \binom{n}{n-k} \).

**Proof.** The two theorems above follow directly from definition 1.0.1.

**Theorem 1.0.6.** For all \( n \geq k \geq 1 \),

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}
\]

**Proof.** We find a problem that can be solved in two ways to give us the two answers. We know what \( \binom{n}{k} \) counts, so that will be our problem. Consider the following question: How many subsets of size \( k \) are there from a set of \( n \) elements? The first answer is \( \binom{n}{k} \). For the second answer, consider the two cases: whether the first element is included in the subset, or it is not.

Case 1: The first element is included in the subset of size \( k \). In this case, we can just count the number of ways to choose a subset of size \( (k-1) \) from the set of \( (n-1) \) elements. This number is \( \binom{n-1}{k-1} \).
Case 2: The first element is not included in the subset of size $k$. In this case, we count the number of ways to choose a subset of size $k$ from the set of $(n - 1)$ elements. This number is $\binom{n - 1}{k}$.

Since the two cases cover all the possible ways to choose a subset of size $k$, and since they are disjoint, the answer must be the sum of the two. Compared to our first answer, it follows that 

$$\binom{n}{k} = \binom{n - 1}{k - 1} + \binom{n - 1}{k}.$$ 

This type of proof is called a combinatorial proof. It is a very powerful tool that is most often used to prove otherwise complicated algebraic identities.

**Theorem 1.0.7.** For all $n \geq k \geq 1$,

$$\frac{n + 1}{k + 1} = \sum_{i=k}^{n} \frac{i}{k}$$

**Proof.** By Theorem 1.0.6, 

$$\frac{n + 1}{k + 1} = \frac{n}{k} + \frac{n}{k + 1}.$$ 

Applying Theorem 1.0.6 again, we have 

$$\frac{n}{k + 1} = \frac{n - 1}{k} + \frac{n - 1}{k + 1}.$$ 

Continue this process, we have 

$$\frac{n + 1}{k + 1} = \frac{n}{k} + \frac{n - 1}{k} + \frac{n - 2}{k} + \cdots + \frac{k}{k}.$$ 

This result is also known as the Hockey-stick identity. Binomial coefficients can be represented like the Pascal triangle, and when we add every term in this sum to get $\frac{n + 1}{k + 1}$, the result will look like a hockey stick.
This is the last result that we will need in order to derive a formula for \( \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j \).

**Theorem 1.0.8.** For all \( n > j \geq 1 \),

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j = \frac{n + j}{j + 1}.
\]

**Proof.** We will prove by induction. Let \( P(j) \) be the statement that for \( n \geq j \geq 1 \),

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j = \frac{n + j}{j + 1}.
\]

First we check \( P(1) \): \( \sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{n+1}{2} \), which is true. Now suppose \( P(k) \) is true for some \( k \), that is,

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_k=1}^{a_{k-1}} a_k = \frac{n + k}{k + 1}.
\]

We will show that \( P(k+1) \) is also true. We can write

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_{k+1}=1}^{a_{k+1}-1} a_{k+1} = \sum_{j=1}^{n} j + k = \frac{n + k + 1}{k + 2}
\]

by Theorem [1.0.7] giving us the desired result. \( \square \)

Using a combinatorial tool, we found a way to represent a very messy nested sum as a much simpler expression. In the next chapters, we will look at some common applications of the binominal coefficients, and explain how the results can be interpreted as a nested sum of integers.
Chapter 2

Graphical Representation

2.1 The Single Sum

Perhaps the easiest way to see why these sums are what they are is to draw a figure. Let’s consider the single sum $\sum_{i=1}^{n} i$. The following two figures show us why this sum is $\frac{n(n + 1)}{2}$.

(a) Dots Representation

(b) Block Representation
The two figures are very much self-explanatory. However, these are not the only ways to prove the formula. In discrete mathematics, or specifically graph theory, it is often useful to represent objects by vertices, and relationships between objects by edges adjoining corresponding vertices. The following example is a weighted directed graph from a point S to a point T. This can be understood as a water distribution map from a source S to a destination T, with capacities indicated by the weights on the edges. Since the water is not escaping the system, the total output at S must equal the total input at T, i.e.,

\[(5)(6) = 2 + 4 + 6 + 8 + 10\]

or equivalently,

\[1 + 2 + 3 + 4 + 5 = \frac{(5)(6)}{2}\]

(c) Aqueduct System

A similar figure in the general case yields the desired formula,

\[1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}\]
2.2 The Double Sums

A figure for the double sum is much more complicated, and we will need another result before we can start constructing it.

**Theorem 2.2.1.** The sum of all odd integers from 1 to $2n - 1$ can be expressed as

$$
\sum_{i=1}^{n} (2i - 1) = n^2.
$$

**Proof.** Here is a proof without words.

This is the result that we will need. Similar to Figure 2.1(a), we will create a visual representation for the double sum, only in 3D. For the sum $\sum_{i=1}^{n} \sum_{k=1}^{i} k$, the first term is $a_1 = 1$, the second term is $a_2 = 1 + 2$, so on and so forth, and the last term is $a_n = 1 + 2 + \ldots + n$. Now we will build a tetrahedron out of this sum, with the last term $a_n$ being the ground floor, the term on the second floor is $a_{n-1}, \ldots$, and the term on the top floor, the $n^{th}$ floor, is $a_1$. We line up these terms in such a way that the number 1 in all the terms $1 + \ldots + k, 1 \leq k \leq n$ are on top of each other:
The purpose of forming a visual representation is to give us a way to understand why the sum is equal to \( \frac{1}{6}n(n + 1)(n + 2) \). To do that, we will “fill up” the tetrahedron to make a prism with the base of \( 1 + 2 + \ldots + n \), and the height of \( n + 2 \). That is, we will “fill up” the tetrahedron in a normal sense, and then we will build another two floors on top of our prism.
Figure 2.1: Forming a Prism by filling in

Let $A$ denote this new prism, then

$$|A| = (1 + 2 + \ldots + n)(n + 2)$$

$$= \frac{1}{2}n(n + 1)(n + 2)$$

Let $S$ denote the double sum $\sum_{i=1}^{n} \sum_{k=1}^{i} k$. According to Theorem 1.0.3, we want to show $|S| = \frac{1}{3}|A|$. Let $T = A - S$, we will show $|S| = \frac{1}{2}|T|$. Note that we can write
the elements of $T$ in descending order of floors as follows:

$$T = 1 + 2 + 3 + \ldots + n$$
$$+ 1 + 2 + 3 + \ldots + n$$
$$+ \quad 2 + 3 + \ldots + n$$
$$+ \quad 3 + \ldots + n$$
$$\quad .$$
$$\quad .$$
$$\quad .$$
$$\quad + \quad (n - 1) + n$$
$$+ \quad n$$

Leaving the first term alone, we can rewrite $T$ as:

$$T = 1 + 2 + 3 + \ldots + n$$
$$+ 1 + 2^2 + 3^2 + \ldots + n^2$$

(2.1)

By Theorem 2.2.1, this is equivalent to:

$$T = 1 + 2 + 3 + \ldots + n$$
$$+ 1$$
$$+ 1 + 3$$
$$+ 1 + 3 + 5$$
$$\quad .$$
$$\quad .$$
$$\quad .$$
$$\quad + 1 + 3 + \ldots + (2n - 1)$$
On the other hand, \( S = \sum_{i=1}^{n} \sum_{k=1}^{i} k \), so

\[
2S = 2 + 2 + 4 + 2 + 4 + 6 + \ldots \ldots + 2 + 4 + \ldots + 2n
\]

\[
2S = (1) + 1 + (1 + 3) + 2 + (1 + 3 + 5) + 3 + \ldots \ldots + (1 + 3 + 5 + \ldots + (2n - 1)) + n
\]

\[
2S = T
\]

We have shown that \( S = \frac{1}{2} T \), which implies \( S = \frac{1}{3} A \), and therefore have verified the formula \( \sum_{i=1}^{n} \sum_{k=1}^{i} k = \frac{1}{6} n(n + 1)(n + 2) \).

This method, unfortunately, will not give us a representation for the Triple Sum and beyond that because it would require more than three dimensions. However, one can conjecture that the m-multiple sums can be represented as a hyperpyramid in a hypercube of \((m + 1)\) dimensions with lengths \(n, n + 1, n + 2, \ldots n + m + 1\).
Chapter 3

Counting Proofs

In the previous chapters we came up with the closed form formula for the multiple sums of consecutive integers, namely \( \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} ... \sum_{a_j=1}^{a_{j-1}} a_j = \frac{n + j}{j + 1} \). In this chapter we will look at some common application of the choose function, or the binomial coefficient, and come up with a way to represent them as the sum of integers. In other words, we will prove the formula again using different combinatorial counting methods.
3.1 Counting Lattice Paths

In this section we will discuss the question: Given an $m \times n$ grid, how many lattice paths are there from $(0, 0)$ to $(m, n)$?

In other words, how many ways can we get from $A$ to $B$ if we are only allowed to go up or right? Since the grid is $m \times n$, a path from $A$ to $B$ would cover a total of $m + n$ steps. Now every path will have to go up $m$ times and to the right $n$ times. So it’s simply a matter of choosing which $m$ steps to go up. Therefore, the total number of paths from $A$ to $B$ is $\frac{m + n}{m}$.

The question is: How can we count all the possible paths from $A$ to $B$ in a way that can be represented as the multiple sums of consecutive integers, so that we once again prove the formula in Chapter 1? We will count it in a way that does not require any binomial coefficient.

**Corollary 3.1.1.** The number of lattice paths on a $1 \times n$ grid is $n + 1$. 


The corollary should be self-evident, as every path has to go up once, and there are only \( n + 1 \) places where it can go up. Now consider an \( m \times n \) grid. We will assign coordinate to each point on the grid with \( A = (0, 0) \) and \( B = (m, n) \). Let \( P(x, y) \) denote the number of paths from \( A \) to the point with coordinate \( (x, y) \). Note that for a path to reach \( B \), it has to either reach \( (n, m - 1) \) or \( (n - 1, m) \) first.

Since we can only go up or right, every path that goes to \( (n, m - 1) \) cannot go through \( (n - 1, m) \), and vice versa. That means the number of paths from \( A \) to \( B \) will be the sum of all the paths from \( A \) to \( (n, m - 1) \) and from \( A \) to \( (n - 1, m) \). In other words, \( P(n, m) = P(n, m - 1) + P(n - 1, m) \). Repeating the same argument, we can see that the number of paths from \( A \) to \( (n - 1, m) \) must be the sum of all the paths from \( A \) to \( (n - 1, m - 1) \) and from \( A \) to \( (n - 2, m) \), so \( P(n - 1, m) = P(n - 1, m - 1) + P(n - 2, m) \). Applying the same argument repeatedly, we have \( P(n, m) = \sum_{i=1}^{n+1} P(i, m - 1) \). The sum goes to \( n + 1 \) because an \( m \times n \) grid gives
$n+1$ horizontal nodes in each row.

But every path from $A$ to each of these points on the $m-1$ row can also be represented as the sum of all the paths from $A$ to every point on the $m-2$ row up to that point. That is, $P(n,m-1) = \sum_{i=1}^{n} P(i,m-2)$. Proceed inductively, we arrive at $P(n,m) = \sum_{a_1=1}^{n+1} \sum_{a_2=1}^{a_1} \ldots \sum_{a_{m-1}=1}^{a_{m-2}} a_{m-1} = \frac{n+1+m-1}{m-1+1} = \frac{n+m}{m}$, confirming the equality.

### 3.2 Number of monotonic functions

In this section we will discuss the question: How many increasing functions are there from a set $[n] = \{1,2,...,n\}$ to a set $[k] = \{1,2,...,k\}$? Once again we will come up with a combinatorial proof by counting in two different ways.

**Definition 3.2.1.** A **multiset** is a generalization of the concept of set that allows multiple instances of the multiset’s elements. The length of a multiset is the number of elements (not necessarily distinct) of that multiset.

**Example 3.2.1.** Let $A = \{1,2,3\}$, an example of a multiset of length 5 of $A$ is $B = \{1,1,1,3,3\}$. 
Definition 3.2.2. The number of multisets of length $k$ of a set of $n$ elements is called $n$ multichoose $k$, denoted $\binom{n}{k}$. The quantity $n$ multichoose $k$ is given by the formula

$$\binom{n}{k} = \frac{n + k - 1}{k}$$

Remark 1. Multichoose problems are often called “stars and bars” problems. For example, suppose a recipe called for 5 pinches of spice, out of 9 spices. Each possibility is an arrangement of 5 spices (stars) and 9-1 dividers between categories (bars), where the notation $**|||*|*|||*$ indicates a choice of spices 1, 1, 5, 6, and 9 (Feller 1968, p. 36). The number of possibilities in this case is then

$$\frac{9}{5} = \frac{9 + 5 - 1}{5} = 1287.$$

Theorem 3.2.1. The number of monotonic functions from $[n]$ to $[k]$ is $\binom{k}{n}$.

Proof. Since each element of $[n]$ must be mapped to an element of $[k]$, we can list all the elements of $[n]$ in a straight line as $1, 2, ..., n - 1, n$ and put $k - 1$ dividers between them to indicate the output. For example $**...*||...*$ means every element from 1 to $n - 1$ is mapped to 1 and $n$ is mapped to $k$. Thus the number of monotonic functions from $[n]$ to $[k]$ is the number of multisets of length $n$ on $[k]$. In other words, this number is $\binom{k}{n}$.

We are more interested in proving the multisum equality that we obtained in Chapter 1, so we will conduct a different way of counting these functions to show the
equality. Consider the following diagram:

Let $S(n, k)$ denote the number of monotonic functions from $[n]$ to $[k]$. Suppose $n$ is mapped to $k$. Then $n - 1$ can be mapped to any element of $[k]$, and the number of monotonic functions that we have in this case is $S(n - 1, k)$. Now suppose that $n$ is mapped to $k - 1$. Then $n - 1$ cannot be mapped to $k$ and thus it only has $k - 1$ available outputs. The number of monotonic functions in this case is $S(n - 1, k - 1)$. Proceeding inductively, we have $S(n, k) = \sum_{i=1}^{k} S(n - 1, i)$. Now for each of $S(n - 1, i)$, we can apply the same argument again. In other words, if $n - 1$ is mapped to $j$, for some $1 \leq j \leq i$, there will be $S(n - 2, j)$ monotonic functions.
Thus $S(n - 1, i) = \sum_{j=1}^{i} S(n - 2, j)$. Repeat this process until the domain is $[1]$. Now the number of monotonic function from $[1]$ to $[k]$ is $k$ because there are only $k$ possible outputs for 1. Thus the total number of monotonic functions from $n$ to $k$ is $S(n, k) = \sum_{a_1=1}^{k} \sum_{a_2=1}^{a_1} \ldots \sum_{a_{n-1}=1}^{a_{n-2}} a_{n-1}$. But we know this number is also $\frac{k}{n}$. Therefore $\sum_{a_1=1}^{k} \sum_{a_2=1}^{a_1} \ldots \sum_{a_{n-1}=1}^{a_{n-2}} a_{n-1} = \frac{k}{n} = \frac{k + n - 1}{n}$, confirming the desired equality.
Chapter 4

Sum of higher degree integers

In the previous chapters we have looked at the multiple sum of consecutive integers. In this chapter we will discover a pattern for higher power of integers, namely, a closed form formula for

\[ \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j^k \]

4.1 Sum of squares

**Theorem 4.1.1.** Let \( n \) be a positive integer, then

\[ \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n + 1)(2n + 1) \]

*Remark 2.* The above result is well-known and we will not be proving it here. However, our goal is to find the multiple sum of squares, and it will become messy very quickly if we leave it in this form. Instead, we will try to convert it into binomial coefficient form.

**Theorem 4.1.2.** Let \( n \) be a positive integer, then

\[ \sum_{k=1}^{n} k^2 = \frac{n + 1}{3} + \frac{n + 2}{3} \]
Proof. By Theorem 4.1.1 \( \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1) = \frac{1}{6} n(n+1)(n-1+n+2) = \frac{1}{6} (n-1)n(n+1) + \frac{1}{6} n(n+1)(n+2) = \frac{n+1}{3} + \frac{n+2}{3} \). 

This is the form that we will use to prove the multiple sum of squares, because the Hockey Stick theorem makes everything a lot easier.

**Theorem 4.1.3.** Let \( n \) be a positive integer, then

\[
\sum_{i=1}^{n} \sum_{k=1}^{i} k^2 = \frac{n+2}{4} + \frac{n+3}{4}
\]

Proof. By the Hockey Stick Theorem, and by Theorem 4.1.2

\[
\sum_{i=1}^{n} \sum_{k=1}^{i} k^2 = \sum_{i=1}^{n} \frac{i+1}{3} + \frac{i+2}{3} = \frac{n+2}{4} + \frac{n+3}{4}
\]

It should be sufficient to see a pattern now, and that is what we will do next.

**Theorem 4.1.4.** Let \( n \) be an integer, then

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_k=1}^{a_{k-1}} a_k^2 = \frac{n+k}{k+2} + \frac{n+k+1}{k+2}
\]

Proof. We will use induction. Let \( n \) be an integer. Let \( P(k) \) be the statement that \( \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_k=1}^{a_{k-1}} a_k^2 = \frac{n+k}{k+2} + \frac{n+k+1}{k+2} \). Base case: \( P(1) : \sum_{k=1}^{n} k^2 = \frac{n+1}{3} + \frac{n+2}{3} \), which is true. Now suppose \( P(i) \) is true for some \( i \), we will show that \( P(i+1) \) is also true. But by the inductive hypothesis, \( \sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_k=1}^{a_{k+1}} a_k^2 = \sum_{a_1=1}^{n} a_1 + k + \frac{a_1 + k + 1}{k+2} = \frac{n+k+1}{k+3} + \frac{n+k+2}{k+3} \), by the Hockey Stick Theorem, completing the induction proof.

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Remark 3. The problem could have been very complicated with a traditional algebra approach. However, binomial coefficient and combinatorial proofs make it much more simple.

4.2 Sum of Cubes and Higher Powers

For this section we will attempt to find a formula for the sum \( \sum_{k=1}^{n} k^i \), for \( i \geq 3 \). For the powers of 3, 4, 5, we actually have a closed form algebraic formula for these sums. They are: \( \sum_{k=1}^{n} k^3 = \frac{1}{4} n^2 (n+1)^2 \), \( \sum_{k=1}^{n} k^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1) \), and \( \sum_{k=1}^{n} k^5 = \frac{1}{12} n^2 (n+1)^2 (2n^2+2n-1) \). However, in order to look at the multiple sums of higher power, we will need a way to transform these expressions into binomial coefficients. In order to do this, we will look at the Pascal Triangle again.

\[
\begin{array}{cccccccccccc}
1 & & & & & & & & & & & & \\
1 & 1 & & & & & & & & & & & \\
1 & 2 & 1 & & & & & & & & & & \\
1 & 3 & 3 & 1 & & & & & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & & & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & & & & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & & & \\
1 & 7 & 21 & 35 & 35 & 21 & 7 & 1 & & & & & \\
1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 & 1 & & & & \\
1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 & 9 & 1 & & & \\
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & 10 & 1 & \\
\end{array}
\]

Label the diagonals of the Pascal Triangle in the following manner: Start from the top left corner and go downward to the right. The zeroth diagonal is \( 1 - 1 - 1 - ... \), the first diagonal is \( 1 - 2 - 3 - ... \), the second diagonal is \( 1 - 3 - 6 - ... \), so on and so forth. What does the sum \( \sum_{k=1}^{n} k \) look like on the Pascal triangle? It is just the
The sum along the first diagonal, and we know \( \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{n+1}{2} \). Where is \( \frac{n+1}{2} \)? It is at the intersection of the second diagonal and the \((n+1)^{th}\) row.

It should be clear the sum \( \sum_{k=1}^{n} k = \frac{n+1}{2} \) is just a simple case of the Hockey Stick theorem. With that in mind, what can we say about the sum \( \sum_{i=1}^{n} \sum_{k=1}^{i} k \)? This double sum is the sum of the previous sums of integers, so it is the sum of every term along the second diagonal up to the \((n+1)^{th}\) row. We can guess that the value of this double sum will be at the intersection of the third diagonal and the \((n+2)^{th}\) row. From Theorem 1.0.3 \[\sum_{i=1}^{n} \sum_{k=1}^{i} k = \frac{1}{6}n(n+1)(n+2) = \frac{n+2}{3},\]
confirming our guess. In general, if we have \(k\)-multiple sums, the result will be on the \((k+1)^{th}\) diagonal and on the \((n+k)^{th}\) row. This behavior will be important when we look at the sum of higher power.

Next we will revisit the sum of squares. Consider the question: Can you find the squares in the Pascal Triangle? From a quick look, we cannot find any perfect square in it. However, note that the second diagonal starts with 1-3-6-..., and 1 = 0 + 1, 4 = 1 + 3, 9 = 3 + 6, 16 = 6 + 10, so on and so forth. That is, we can form perfect squares by adding any two consecutive terms on the second diagonal.

Now the sum of squares is the sum of all these sums of two consecutive terms. A natural question to ask is whether what we get is also a sum of two consecutive terms, and the answer is yes. By Theorem 4.1.2 \[\sum_{k=1}^{n} k^2 = \frac{n+1}{3} + \frac{n+2}{3},\]
which is the sum of two consecutive terms on the third diagonal, on the \((n+1)^{th}\) and \((n+2)^{th}\) rows. Similarly, the multiple sums of squares is simply a sum of two consecutive terms on the diagonal that is shifted downward based on the number of the sums. At this point we already have the sufficient tools to tackle the sum of
higher powers. We just need to define one more term to make the argument easier.

**Definition 4.2.1.** A block of length \( n \) on the Pascal Triangle is a sequence of \( n \) consecutive integers on a diagonal. A weighted block is a sequence where each value is multiplied by its corresponding weight.

**Example 4.2.1.** A few blocks of length 2 on the second diagonal would be \( \{0, 1\}, \{1, 3\}, \{3, 6\}, \ldots \). Note that the total value of each of these blocks is a perfect square, and the sum of squares is just the sum of these 2-blocks. As discussed earlier, the result is also a 2-block, shifted down by 1 diagonal. Since we only care about the sum of all the elements in a block, we will just say the value of the block to imply the sum of all of its elements.

**Example 4.2.2.** A weighted block of size \( a_1 - a_2 - \ldots - a_n \) means a block of length \( n \) where the \( i^{th} \) element has weight \( a_i \). For example, a weight block of size \( 1 - 2 - 3 \) is a block of length 2, where the first element has weight 1, the second element has weight 2, and the last element has weight 3. A few weighted blocks of size 1-2-3 on the first diagonal would be \( \{0, 0, 1\}, \{0, 2, 6\}, \{1, 4, 9\}, \ldots \). Now if we add the values of these blocks and then add them together, would the result also be a 1-2-3 block, but on the second diagonal? The following theorem will give the answer.

**Theorem 4.2.1. The Generalized Hockey Stick theorem.** The Hockey Stick Theorem holds for any weighted block of any length. That is, the sum of weighted blocks along a diagonal will result in the same weighted block shifted down one diagonal.

**Proof.** The sum of the weighted blocks is the sum of all the corresponding weighted element in each block, and the sum of each element follows the Hockey Stick Theorem. Thus the resulting sum has the same structure as the original block. \( \Box \)
Remark 4. The Generalized Hockey Stick Theorem is the last tool that we will need to study the sum of higher powers. As a result of this theorem, if we can express the powers of integers as some weighted block, we will immediately have all the closed form formula for the multiple sums of powers of integers.

The natural question to ask is: Can we express any power of integers as a sum of some weighted block? The answer is yes. For smaller powers, it is not too difficult to find a general pattern for the blocks. The following theorems can be easily verified by computational machine or algebraic method. The latter is a direct result of the former, using the Generalized Hockey Stick Theorem. We will not, however, spend too much time proving the formula. Instead, we will discuss the significance of the coefficients needed for each power.

**Theorem 4.2.2.** For all $n \in \mathbb{N}$,

$$n^3 = \frac{n}{3} + 4 \frac{n+1}{3} + \frac{n+2}{3}$$

$$n^4 = \frac{n}{4} + 11 \frac{n+1}{4} + 11 \frac{n+2}{4} + \frac{n+3}{4}$$

$$n^5 = \frac{n}{5} + 26 \frac{n+1}{5} + 66 \frac{n+2}{5} + 26 \frac{n+3}{5} + \frac{n+4}{5}$$

**Theorem 4.2.3.** For all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} k^3 = \frac{n+1}{4} + 4 \frac{n+2}{4} + \frac{n+3}{4}$$

$$\sum_{k=1}^{n} k^4 = \frac{n+1}{5} + 11 \frac{n+2}{5} + 11 \frac{n+3}{5} + \frac{n+4}{5}$$

$$\sum_{k=1}^{n} k^5 = \frac{n+1}{6} + 26 \frac{n+2}{6} + 66 \frac{n+3}{6} + 26 \frac{n+4}{6} + \frac{n+5}{6}$$
The proof of these theorems are left to the reader. The first theorem says that we can express the third, fourth, fifth powers of integers on the third, fourth, fifth diagonal, respectively. If we take a weighted block of size 1-4-1 on the third diagonal and add up all the terms, we will get a perfect cube. For example, \(4 + 4(10) + 20 = 64 = 4^3\), and \(10 + 4(20) + 35 = 125 = 5^3\). The second theorem says that the sum of these power has the same block setup as the terms. In other words, it is the same weighted block, shifted down one diagonal by the Generalized Hockey Stick Theorem. Notice that the sum of the coefficients for the \(k^{th}\) power is \(k!\), and the coefficients look like they represent a variation of the Pascal triangle. That observation is true. The coefficients follow a pattern called the Eulerian Numbers, and here is what the first few terms of the Eulerian Number Triangle:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
1 & 4 & 1 \\
1 & 11 & 11 & 1 \\
1 & 26 & 66 & 26 & 1 \\
1 & 57 & 302 & 302 & 57 & 1 \\
1 & 120 & 1191 & 2416 & 1191 & 120 & 1 \\
\end{array}
\]

**Definition 4.2.2.** The Eulerian number \(\binom{n}{k}\) is the number of permutations of the numbers 1 to \(n\) in which exactly \(m\) elements are greater than the previous element.

**Example 4.2.3.** \(\binom{3}{1}\) is the number of permutations of \(\{1,2,3\}\) in which exactly 1 elements are greater than the previous elements. All the permutations of \(\{1,2,3\}\) are: 123,132,213,231,321,312. Note that 132,213,231,312 satisfy this condition. Thus \(\binom{3}{1} = 4\), which corresponds to the middle number of the third row of the Eulerian Number Triangle.

**Theorem 4.2.4.** Worpitzky’s Identity. For all \(m \in \mathbb{N}, a \in \mathbb{R}\),

\[
a^m = \sum_{k=0}^{m-1} \binom{m}{k} (a+k)^m
\]
We will not be too concerned with how this type of permutation is related to the higher power of integers. All we need to know is that Worpitzky’s Identity gives a nice way to express any power of integer as a sum of binomials. Note that as \( k \) goes from 0 to \( n \), the values taken by the elements of \( \binom{x+k}{n} \) become a weighted block of length \( n \) on the \( n^{th} \) diagonal. If we consider \( x \) to be integers from 1 to \( N \) and sum up every term, we get what we have discovered so far: a sum of weighted block along a diagonal, which is also a weighted block of the same size on the next diagonal. Moreover, we can express this sum for any power of \( x \)! The following theorem follows directly from Worpitzky’s Identity.

**Theorem 4.2.5.** For all \( n, m \in \mathbb{N} \),

\[
\sum_{a=1}^{n} a^m = \sum_{k=0}^{m-1} k \binom{n+k+1}{m+1}
\]

But that is not all. We already know that if we put another sum in front of the summation, all we do is adding all the weighted block of the corresponding diagonal, and the result will be a weighted block that is shifted down two diagonal from the original terms. Repeat inductively, we have a closed form formula for any \( k \)-multiple sums of any power of consecutive integers.

**Theorem 4.2.6.** For all \( n, m \in \mathbb{N} \),

\[
\sum_{a_1=1}^{n} \sum_{a_2=1}^{a_1} \ldots \sum_{a_j=1}^{a_{j-1}} a_j^m = \sum_{k=0}^{m-1} k \binom{n+k+j}{m+j}
\]

**Remark 5.** All of this are possible because of the Generalized Hockey Stick Theorem, and because we could come up with a way to express \( x^n \) as a sum of terms along a diagonal. Using the same concept, we can find the multiple sums of many other sequences, as long as we have a way to express the terms on the diagonal of the Pascal Triangle.
Example 4.2.4. As a direct result of Theorem 4.2.6 we have:

\[
\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} k^4 = \frac{n + 3}{7} + 11 \frac{n + 4}{7} + 11 \frac{n + 5}{7} + \frac{n + 6}{7}
\]

\[
\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} \sum_{m=1}^{k} m^5 = \frac{n + 4}{9} + 26 \frac{n + 5}{9} + 66 \frac{n + 6}{9} + 26 \frac{n + 7}{9} + \frac{n + 8}{9}
\]

4.3 Other Sequences

Consider a weighted block of length 3 and of weight 1-1-1 on the second diagonal. Let us compute the first few terms: 1, 1+3=4,1+3+6=10,3+6+10=19,... What sequence starts with 1,4,10,19,...? It is the centered triangular number sequence.

Definition 4.3.1. The centered polygonal numbers are a class of series of figurate numbers, each formed by a central dot, surrounded by polygonal layers with a constant number of sides. Each side of a polygonal layer contains one dot more than the previous layer.

Example 4.3.1. Here are the first few centered triangular numbers: 1,4,10,19,...

![Centered Triangular Numbers](image)

Theorem 4.3.1. The \(n^{th}\) centered \(k\)-gonal number is given by the formula

\[
C_{k,n} = \frac{kn}{2}(n - 1) + 1
\]
A computational machine or algebra method can be used to show that the formula for the centered triangular number given above agrees with the sum of the 1-1-1 block on the second diagonal. Once again we will not be too concerned with deriving the formula. Instead we will try to find as many sequences as possible that can be expressed by terms along the diagonal.

Also on the second diagonal, let us consider a weighted block of length 3 and of weight 1-2-1. The first few terms are 1, 5, 13, 25, ... This sequence is known as the centered square numbers.

This behavior should naturally beg the question: Would a weighted block of size 1-3-1 gives the sequence for the centered pentagonal numbers? The first few terms are 1, 6, 16, 31, ... Indeed this is the centered pentagonal numbers sequence. And this pattern holds true for all blocks of size 1-k-1 on the second diagonal.

**Theorem 4.3.2.** The centered \((k + 2)\)-gonal numbers can be express by weighted blocks of size \(1 - k - 1\) on the second diagonal of the Pascal Triangle.

A strictly algebraic proof is very straightforward and is left to the reader.

There are many more sequences that we can find on the Pascal. On the third diagonal alone: the 1-1-1 block gives the Magic Constant sequence, the 1-2-1 block gives the octahedral number sequence, and the 1-3-1 block gives the sequence of numbers of atoms in a decahedron with \(n\) shells,... On the fourth diagonal: the 1-1-1 block gives the doubly triangular number sequence, the 1-2-1 block gives the 4-dimensional centered polygonal number sequence,... And who knows what else we can find. For sequences that we can express as weighted blocks, we immediately have a way to convert the sequence into a nice closed form of sum of binomial coefficients. With this closed form, finding the sums and multiple sums of the sequence becomes a lot easier, all thanks to the Generalized Hockey Stick Theorem.
Bibliography

